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On the New Algorithm of Testing and Comparing Size Corrected Powers for Testing Multivariate Normality

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***Abstract*— Parametric models are mainly based on univariate or multivariate normality assumptions. Uniformly most powerful (UMP) test is not available to test multivariate normality. In such a situation, optimal test can be used. But, a very few literature is available on the size corrected power comparison of different multivariate normality tests. In this paper, we propose an algorithm to compare the size corrected powers for testing univariate or multivariate normality. The algorithm can be applied to any existing univariate and multivariate tests, which is the most attractive feature of the proposed new algorithm. We also propose a Cholesky decomposition of the variance-covariance matrix based test, which is simpler than the existing test. Our Monte Carlo simulation study indicates that our proposed and existing tests perform equally in terms of power properties.**

***Keywords*— Cholesky decomposition, UMP test, Optimal test, Monte Carlo.**

I. INTRODUCTION

Most of the parametric models are based on certain assumptions and semi-parametric models based on relaxed assumptions. In most of the cases, errors are assumed to be distributed as univariate or multivariate normal. Violation of the assumption under which a model is constructed, analysis can be distorted in determining the confidence interval, forecast, and testing hypothesis. Therefore, choosing the appropriate test to capture normality is the most important concern of this paper. A very powerful transformation called Box-Cox transformation can be used when the model violates the assumption of normality to make the data usable for making valid statistical decision. Uniformly most powerful (UMP) does not always exist. In such case, we prefer to use an optimal test instead of UMP test. To compare the power of the test, we use simulated size corrected power. The algorithm of size corrected power for testing multivariate normality is very limited. So in this paper, our main objective is to develop the algorithm to calculate size corrected powers of competitive tests, which can be used to find optimal test.

Multivariates are dependent in nature which is complicated for the algebraic treatment but different transformation can be used to transform dependent to independent characteristics such as spectral decomposition, Cholesky decomposition etc. Cholesky decomposition is simpler than Spectral decomposition. In this paper we also propose Spectral type decomposition for multivariate test. The Objective of the Study are i) to develop new test for testing multivariate normality, ii) to develop an efficient algorithm for calculating the size corrected power of the test which can be used to compare the efficiency of the different test, iii) to compare the performance of bivariate and multivariate normality testing procedures by using different decompositions with the new algorithm.

II. EXISTING TESTS

Different univariate and multivariate normality tests are available. Some of them are discussed in the following paragraph.

A. Univariate Normality Tests

Bowman and Shenton Test

Let $(x_1, x_2, \ldots, x_n)$ be an independent observations on a one-dimensional random variable with mean μ and variance σ^2 where $\mu_i = E(X-\mu)^i$ and $\sigma^2 = \mu_2$. Then skewness and kurtosis are defined as follows:

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} \text{ and } \beta_2 = \frac{\mu_4}{\mu_2^{\ 2}}$$

Skewness refers to the symmetry of distribution. For a symmetry distribution like the normal $\sqrt{\beta_1} = 0.$ A distribution that is skewed to the right has $\sqrt{\beta_1} > 0$ while one that is skewed to the left has $\sqrt{\beta_1} < 0$.

Kurtosis refers to the flatness or 'peak ness' of a distribution. The normal distribution has $\beta_2 = 3$ and is used as to reference for other distribution. A leptokurtic distribution is one that is more peaked and heavier tails than the normal, resulting in $\beta_2 > 3.$ A platykurtic distribution has a flatter distribution with shorter tails than the normal, Hence $\beta_2 < 3.$

The sample counterparts are defined as

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i\ ,\quad m_i = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2$$

$$\sqrt{b_1} = \frac{m_3^2}{m_2^3} \quad \text{and} \quad b_2 = \frac{m_4}{m_2^2}$$

Bowman and Shenton (1975) consider the test statistic

$$E^a{}_d = \frac{n\left(\sqrt{b_1}\right)^2}{6} + \frac{n(b_2-3)^2}{24} \sim \kappa^2(2)$$

Jarque Bera Test

The test statistic JB of Jarque Bera is defined by

$$JB = \frac{n}{6}.\left(S^2 + \frac{(k-3)^2}{4}\right)$$

Where the sample skewness $S = \hat{\mu}_3 / \hat{\mu}_2^{3/2}$ is an estimator of $\beta_1 = \mu_3 / \mu_2^{3/2}$ and the sample kurtosis $K = \hat{\mu}_4 / \hat{\mu}_2^{2}$ an estimator of $\beta_2 = \mu_4 / \mu_2^2$, μ_2 and μ_3 are the theoretical second and third central moments, respectively and n is the sample size with its estimates

$$\hat{\mu}_j = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^j, \quad j = 2,3,4.$$

JB is asymptotically chi-squred distributed with two degree of freedom because JB is just the two asymptotically independent standard normal, (see Bowman and Shenton (1975). That means H_0 has to be rejected at level α if $JB \geq \chi^2{}_{1-\alpha,2}$.

Also, Fisher's Cumulant Test**,** Shapiro-Wilk test, Kolmogorov-Smirnov Test**,** Kuiper test, Cramer-von Mises test, Geary's test, Modified EDF test, D'Agostino tests can be used for testing univariate normality.

B. Multivariate Normality Tests

Unfortunately, in practice, testing for multivariate normality is more difficult than univariate normality and relatively few formal methods are available in this context. The most commonly used multivariate normality test is marginal decomposition based Multivariate Omnibus Test which is discussed below.

Let $X' = (X_1, \dots, X_n)$ be a $p \times n$ matrix of n observations on a p -dimensional vector with sample mean and covariance $\bar{X} = n^{-1}(X_1 + \dots + X_n)$ and $S = n^{-1}\breve{X}\breve{X}$ where $\breve{X}' = (X_1 - \bar{X}, \dots, X_n - \bar{X})$.

Create a matrix with reciprocals of the standard deviation on the diagonal:

$$V = diag\left(S_{11}^{-1/2}, \dots, S_{pp}^{-1/2}\right),$$

And form the correlation matrix $C = VSV$. Define the $p \times n$ matrix of transformed observations:

$$R' = H\Lambda^{-1/2}H'V\breve{X}',$$

with $\Lambda = diag(\lambda_1, \dots, \lambda_n)$, the matrix with the eigen values of C on the diagonal. The columns of H are the corresponding eigenvectors, such that $H'H = I_P$ and $\Lambda = H'CH$. Using the population values for C and V, a multivariate normal can thus be transformed into independent standard normal; using sample values this is only approximately so. Using each of the transformed vector $n-$ vectors of observations, we may compute univariate skewness and kurtosis, defining $B_1' = \left(\sqrt{b_{11}}, \dots, \sqrt{b_{1p}}\right)$, $B_2' = (b_{21}, \dots, b_{2p})$ and l as a $p-$ vector of ones, the test statistic:

$$\frac{nB_1'B_1}{6} + \frac{n(B_2 - 3l)'(B_2 - 3l)}{24} \tilde{a}\, \chi^2(2p),$$

where $Z_1' = (z_{11}, \dots z_{1p})$ and $Z_2' = (z_{21}, \dots z_{2p})$.

C. Spectral and Cholesky decomposition based Normality Tests

Along with available multivariate normality tests, Cholesky decomposition and Spectral decomposition based multivariate normality tests are considered here as well. Moreover, Cholesky decomposition has computer built in function for easier use and is much simpler than Spectral decomposition. We propose to use Cholesky decomposition instead of Spectral decomposition to test multivariate normality.

III. Proposed Algorithm of Determining Size Corrected Power

Power calculation for normality test considered by many authors and in most of the cases their suggested approach is based on the percentage of rejection which does not provide size corrected power. To calculate size corrected power of multivariate normality test, we propose the following algorithm:

1. Suppose $x_1, x_2, \dots, x_p$ is a random sample from a p-variate multivariate normal population.
2. Sort each variable $x_{(i1)}, x_{(i2)}, \dots, x_{(in)}$ where $(i = 1, 2, \dots, p)$ in ascending order of magnitude.
3. Multiply the upper k % of data; say 5%, 10% by a positive constant $c \geq 1$.
4. Calculate the power on the basis of the hypothesis. The hypothesis can be stated as

   $H_0 : c = 1$ (i.e., the distribution is normal) against

   $H_1 : c > 1$ (i.e., the distribution is non-normal).

Usually powers of normal distribution is calculated on the basis of the contamination of location or scale parameter whether by increasing or decreasing the parameter value but these contaminations cannot make data non-normal. That's why we are considering the characteristics of normal distribution, which are skewness and kurtosis. By contaminating the upper and lower percentages of data, say 10%, 20% or more, we are making them highly skewed or asymmetric by multiplying with a increasing constant k where $k = 1, 2, 3, \dots$ when $k = 1$ then it will calculate the power of null hypothesis as the value of k will increase it will go far from the null which expresses the departure from normality.

IV. SIMULATION STUDY AND RESULTS

Algorithm on which this study based is enumerated below. The null hypothesis is as follows:

H_0 : *Observations are normally distributed*

H_1 : *Observations are not normally distributed*

To evaluate whether size or level of test achieves advertised α, generate data under normality assumption and calculate proportion of rejections of H_0. To calculate power, we follow the 4 steps proposed above.

A. Power Comparison among Different Univariate Normality Tests

This section compares powers of different univariate normality tests discussed above. In this regard, we generate data for different sample sizes under null hypothesis and carry out 10,000 repetitions to calculate size corrected powers of normality tests with contamination.

Table 1: Powers of Different Univariate Normality Tests with contamination for sample size $n = 50$ *and* 100 with 10000 repetitions.

Sample Size, n	Powers of Univariate Normality Test					Sample Size, n	Powers of Univariate Normality Test				
	BS	JB	FIS	$D.S_k$	$D.Ku$		BS	JB	FIS	$D.S_k$	$D.Ku$
	0.050	0.050	0.050	0.050	0.050		0.050	0.050	0.050	0.050	0.050
	0.060	0.060	0.055	0.089	0.051		0.074	0.074	0.069	0.112	0.065
	0.082	0.082	0.063	0.151	0.048		0.12	0.12	0.101	0.197	0.072
	0.121	0.121	0.077	0.234	0.045		0.196	0.196	0.141	0.327	0.072
	0.176	0.176	0.096	0.336	0.041		0.299	0.299	0.202	0.469	0.065
	0.236	0.236	0.115	0.431	0.035		0.421	0.421	0.289	0.603	0.067
	0.327	0.327	0.147	0.557	0.031		0.556	0.556	0.376	0.728	0.056
	0.415	0.415	0.175	0.652	0.025		0.673	0.673	0.471	0.814	0.471
	0.506	0.506	0.199	0.731	0.199		0.78	0.78	0.563	0.887	0.563
	0.604	0.604	0.252	0.806	0.252		0.863	0.863	0.655	0.938	0.655
	0.679	0.679	0.284	0.865	0.284		0.916	0.916	0.738	0.963	0.738
	0.759	0.759	0.33	0.905	0.33		0.953	0.953	0.815	0.982	0.815
50	0.818	0.818	0.371	0.935	0.371	100	0.975	0.975	0.868	0.99	0.868
	0.87	0.87	0.425	0.958	0.425		0.987	0.987	0.916	0.996	0.916
	0.9	0.9	0.461	0.97	0.461		0.993	0.993	0.941	0.998	0.941
	0.932	0.932	0.5	0.981	0.5		0.997	0.997	0.968	0.999	0.968
	0.951	0.951	0.552	0.989	0.552		0.998	0.998	0.98	1	0.98
	0.967	0.967	0.597	0.991	0.597		0.999	0.999	0.99	1	0.99
	0.977	0.977	0.637	0.996	0.637		1	1	0.992	1	0.992
	0.986	0.986	0.679	0.997	0.679		1	1	0.997	1	0.997
	0.99	0.99	0.721	0.998	0.721		1	1	0.998	1	0.998
	0.994	0.994	0.75	0.999	0.75		1	1	0.999	1	0.999
	0.997	0.997	0.78	1	0.78		1	1	0.999	1	0.999
	0.997	0.997	0.811	1	0.831		1	1	1	1	1
	1	1	1	1	1		1	1	1	1	1

BS = Bowman-Shenton, JB = Jarque-Bera, FIS = Fisher's Cumulant, $D.S_k$ = D'Agostino Skewness, $D.Ku$ = D'Agostino Kurtosis.

B. Power calculation for Multivariate Normal data

This section compares powers of the multivariate normal data using different decompositions with upper and lower contamination of a certain percentage say 10%, 20% or more with their power curves.

From the above power curves, we observed that the powers of the Cholesky decomposition, Spectral decomposition and Doornik Hansen's Marginal Decomposition are same in all of the cases. Since the powers of the multivariate normality tests with all the decompositions are almost close, so we recommend to use cholesky decomposition than the others established decomposed based testing methods because of it's computational convenience and flexibility.

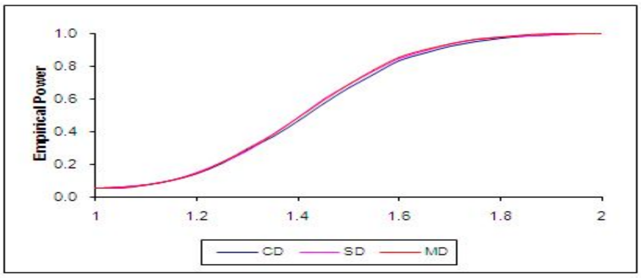

Fig.1: Empirical Powers of multivariate normal data with lower contamination Using Different decompositions for sample size $n = 50$

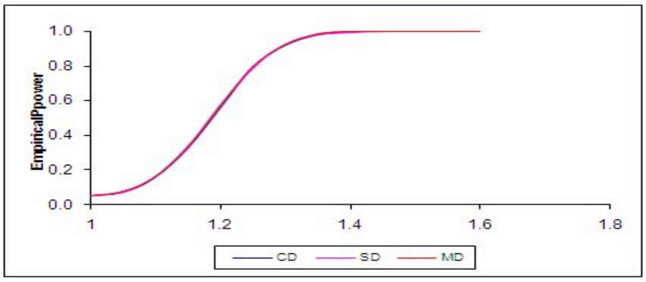

Fig.2: Empirical Powers of multivariate normal data with lower contamination Using Different decompositions for sample size $n = 200$.

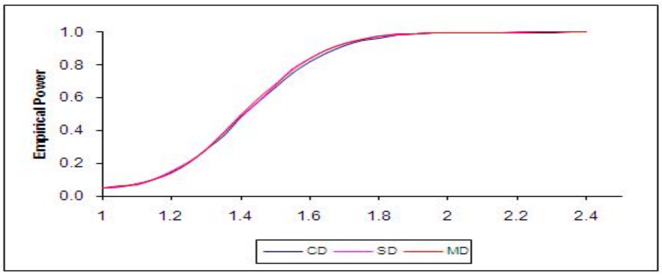

Fig.3: Empirical Powers of multivariate normal data with upper contamination Using Different decompositions for sample size $n = 50$.

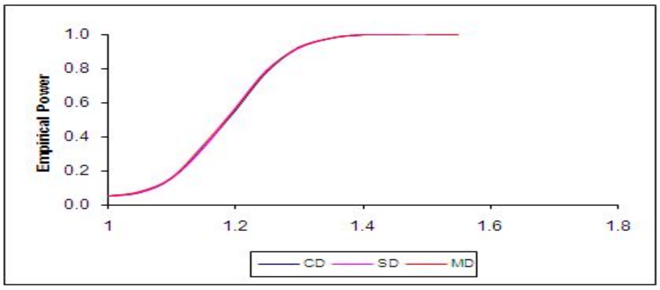

Fig.4: Empirical Powers of multivariate normal data with upper contamination Using Different decompositions for sample size $n = 200$.

Table 2: Lower Contaminated Powers of Multivariate Normality Test Using Different Decompositions or Transformations for Sample Size $n = 50, 100, 200$ *and* 300 with 10000 repetitions.

Sample Size, n	Powers of Multivariate Normality Test Using Different Transformations: CD	SD	MD
50	0.05000	0.05000	0.05000
	0.05690	0.05610	0.05320
	0.07380	0.07470	0.07160
	0.10180	0.10330	0.09750
	⋮	⋮	⋮
	0.99900	0.99910	0.99940
	0.99960	0.99990	0.99990
	0.99980	0.99990	0.99990
	1	0.99990	1
	1	1	1
100	0.05000	0.05000	0.05000
	0.06020	0.06430	0.06020
	0.09540	0.10030	0.09540
	0.16690	0.16570	0.16690
	⋮	⋮	⋮
	0.99330	0.99870	0.99780
	0.99780	0.99950	0.99970
	0.99970	0.99980	1
	0.99980	0.99990	1
	1	1	1
200	0.05000	0.05000	0.05000
	0.07200	0.07350	0.07490
	0.15280	0.16130	0.16150
	0.32130	0.33160	0.33300
	⋮	⋮	⋮
	0.99560	0.99520	0.99640
	0.99930	0.99920	0.99960
	0.99990	0.99990	0.99990
	1	1	1
	1	1	1
300	0.05000	0.05000	0.05000
	0.08760	0.08440	0.15000
	0.22990	0.22970	0.32000
	⋮	⋮	⋮
	0.94390	0.94010	0.96000
	0.99100	0.99200	1
	0.99900	0.99890	1
	1	1	1

CD means Cholesky Decomposition, SD means Spectral Decomposition, MD means Doornik Hansen's Marginal Decomposition.

Table 3: Upper Contaminated Power of Multivariate Normality Test Using Different Decompositions or Transformations for Sample Size $n = 50, 100, 200$ *and* 300 with 10000 repetitions.

Sample Size, n	Powers of Multivariate Normality Test Using Different Transformations: CD	SD	MD
50	0.05000	0.05000	0.05000
	0.05830	0.05640	0.05860
	0.07430	0.07280	0.07080
	0.10300	0.10160	0.09950
	⋮	⋮	⋮
	0.99970	0.99990	0.99990
	0.99980	1	1
	0.99990	1	1
	1	1	1
	1	1	1
100	0.05000	0.05000	0.05000
	0.06040	0.06000	0.06210
	0.09700	0.09810	0.09830
	0.16450	0.16380	0.17010
	⋮	⋮	⋮
	0.99930	0.99930	0.99990
	0.99990	0.99990	0.99990
	1	0.99990	1
	1	1	1
	1	1	1
200	0.05000	0.05000	0.05000
	0.07200	0.07180	0.07620
	0.15110	0.15240	0.15660
	0.33230	0.33280	0.34260
	⋮	⋮	⋮
	0.97810	0.99520	0.97920
	0.99560	0.99920	0.99680
	0.99930	0.99990	0.99950
	1	1	1
	1	1	1
300	0.05000	0.05000	0.05000
	0.08650	0.08110	0.08870
	0.23650	0.22790	0.24410
	⋮	⋮	⋮
	0.94050	0.94230	0.94470
	0.99010	0.99290	0.99110
	0.99900	0.99900	0.99910
	1	1	1
	1	1	1

CD means Cholesky Decomposition, SD means Spectral Decomposition, MD means Doornik Hansen's Marginal Decomposition.

V. CONCLUSIONS

We propose a general algorithm for calculating size-corrected powers of testing univariate and multivariate normality. This algorithm is applicable to all tests of normality. Also, we recommend that to calculate powers of multivariate normality tests implement Cholesky decomposition based multivariate normality test in practical situations just because of its computational convenience and it is well known that cholesky decomposition has computer built in function as well. Moreover, the powers of the multivariate normality test with all the decompositions are almost same.

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